

# Entanglement in SU(2)-invariant quantum systems: The positive partial transpose criterion and others

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We study entanglement in mixed bipartite quantum states which are invariant under simultaneous SU(2) transformations in both subsystems. Previous results on the behavior of such states under partial transposition are substantially extended. The spectrum of the partial transpose of a given SU(2)-invariant density matrix  $\rho$  is entirely determined by the diagonal elements of  $\rho$  in a basis of tensor-product states of both spins with respect to a common quantization axis. We construct a set of operators which act as entanglement witnesses on SU(2)-invariant states. A sufficient criterion for  $\rho$  having a negative partial transpose is derived in terms of a simple spin correlator. The same condition is a necessary criterion for the partial transpose to have the maximum number of negative eigenvalues. Moreover, we derive a series of sum rules which uniquely determine the eigenvalues of the partial transpose in terms of a system of linear equations. Finally we compare our findings with other entanglement criteria including the reduction criterion, the majorization criterion, and the recently proposed local uncertainty relations.

## I. INTRODUCTION

As it was recognized already in the 1930's by some of the founding fathers of modern physics, the notion of entanglement is one of the most intriguing properties of quantum mechanics, distinguishing the quantum world from the classical one [1,2]. Moreover, quantum entanglement is the key ingredient to many if not almost all concepts and proposal in the field of quantum information theory and processing [3]; recent reviews on progress in the theoretical description and analysis of entanglement are listed in Refs. [4–6].

As far as pure states of a quantum system are concerned, the situation is, from a theory point of view, very clear: there are simple and efficient methods to detect and quantify entanglement in a given pure state. One of the most widely used entanglement measures for this case is certainly the von Neumann-entropy of partial density matrices constructed from the full pure-state density matrix [7]. However, the problem of entanglement in mixed states is in general an open one. A mixed state is said to be non-entangled (or separable) if it can be represented as a convex sum of projectors onto non-entangled pure states. In the following we shall concentrate on bipartite systems. As it was noticed by Peres [8], a necessary criterion for mixed state of a bipartite system to be separable is that its partial transpose with respect to one of the subsystems is positive [9]. Subsequently it was shown by the Horodecki family that this condition is also sufficient if the Hilbert space of the bipartite system has dimension  $2 \times 2$  or  $2 \times 3$  [10]. For larger dimensions, inseparable states with positive partial transpose (PPT) exist [11], i.e. the PPT (or Peres-Horodecki) criterion is in general a necessary but not a sufficient one.

More recently, mixed states being invariant under certain joint symmetry operations of the bipartite system have been studied [12]. The probably oldest example

known to the literature of this kind of objects are the Werner states. Here both parties have local Hilbert spaces of the same dimension, and the Werner states are defined by being invariant under *all* simultaneous unitary transformations  $U \otimes U$ . Another important example from this class of states but with an in general much smaller symmetry group are so-called SU(2)-invariant states [14,15]. Here we regard the two subsystems as spins  $\vec{S}_1$ ,  $\vec{S}_2$ , where  $2S_1 + 1$ ,  $2S_2 + 1$  are the dimensions of the corresponding Hilbert spaces. SU(2)-invariant states are defined to be invariant under all uniform rotations  $U_1 \otimes U_2$  of both spins  $\vec{S}_1$  and  $\vec{S}_2$ , where  $U_{1/2} = \exp(i\vec{\eta}\vec{S}_{1/2})$  are transformations corresponding to the same set of real parameters  $\vec{\eta}$  in the representation of SU(2) appropriate for the spin lengths  $S_1$  and  $S_2$  ( $\hbar = 1$ ). Werner states and SU(2) invariant states are identical for  $S_1 = S_2 = 1/2$ , but for larger spin lengths the SU(2)-invariant states have a clearly smaller symmetry group. By construction, SU(2)-invariant states commute with all components of the total spin  $\vec{J} = \vec{S}_1 + \vec{S}_2$ . In particular, for SU(2)-invariant states acting on bipartite Hilbert spaces with dimension  $2 \times N$ , the Peres-Horodecki criterion can be shown to be necessary *and sufficient* [14], i.e. there are no entangled states of this kind with a positive partial transpose.

SU(2)-invariant density matrices arise from thermal equilibrium states of low-dimensional spin systems with a rotationally invariant Hamiltonian by tracing out all degrees of freedom but those two spins. In fact, in the recent years, entanglement in generic quantum spin models has developed to a major direction of research, see, e.g., Refs. [16–20]. Most recently, SU(2)-invariant states were also studied as a model for entangled multiphoton states produced by parametric down-conversion [21].

Most recently, and during the present work was being completed, a preprint by Breuer appeared [22] where SU(2)-invariant states with common spin length,  $S_1 = S_2$

are studied. The approach there is so far restricted to small spin lengths ( $S_1 = S_2 \leq 3/2$ ), but has the merit to allow for an analysis on the sufficiency of the Peres-Horodecki criterion.

In the present work we extend previous results on entanglement properties of SU(2)-invariant states [14] and compare the PPT criterion with other entanglement criteria including the reduction criterion [23,24], the majorization criterion [25], and the local uncertainty relations studied very recently [26,27]. The latter criteria are very readily applied to SU(2)-invariant states, and these considerations provide instructive illustrations of the logical hierarchy of those entanglement criteria.

This paper is organized as follows. In section II we summarize important properties of SU(2)-invariant states under partial transposition and derive a series of additional results which allow to extend previous findings [14] to the case of larger spin lengths. In the following section we apply the above-mentioned other entanglement criteria to SU(2)-invariant density matrices and compare the results with each other. We close with conclusions in the last section.

## II. SU(2)-INVARIANT STATES UNDER PARTIAL TRANSPOSITION

An SU(2)-invariant state  $\rho$  of a bipartite system of two spins  $\vec{S}_1, \vec{S}_2$  has the general form [14]

$$\rho = \sum_{J=|S_1-S_2|}^{S_1+S_2} \frac{A(J)}{2J+1} \sum_{J^z=-J}^J |J, J^z\rangle_{00} \langle J, J^z|, \quad (1)$$

where the constants  $A(J)$  fulfill  $A(J) \geq 0$ ,  $\sum_J A(J) = 1$ . Here  $|J, J^z\rangle_0$  denotes a state of total spin  $J$  and  $z$ -component  $J^z$ . In particular,  $\rho$  commutes with all components of the total spin  $\vec{J} = \vec{S}_1 + \vec{S}_2$ . Obviously the SU(2)-invariant density matrices form a convex set, i.e. with two given SU(2)-invariant states  $\rho_1, \rho_2$  any convex combination  $\lambda\rho_1 + (1-\lambda)\rho_2$ ,  $\lambda \in [0, 1]$ , has the same property. Let us now consider the partial transpose of an SU(2)-invariant state,  $\rho^{T_2}$ , where we take, without loss of generality, the partial transpositions to be performed of the second subsystem describing the spin  $\vec{S}_2$ . Moreover, let us assume that the partial transposition is performed in the standard basis of joint tensor-product eigenstates of  $S_1^z$  and  $S_2^z$ . As shown earlier [14], under these conditions  $\rho^{T_2}$  commutes with all components of the vector  $\vec{K}$  defined by  $K^x = S_1^x - S_2^x$ ,  $K^y = S_1^y + S_2^y$ ,  $K^z = S_1^z - S_2^z$ , and these operators also furnish a representation of su(2),  $[K^\alpha, K^\beta] = i\varepsilon^{\alpha\beta\gamma} K^\gamma$  (using standard notation). We note that the above result relies on the transformation properties of  $\rho^{T_2}$  [14]. The form of the operators  $\vec{K}$  depends on the basis with respect to the partial transposition is performed. For any choice of basis one finds a set of operators  $\vec{K}$  commuting with  $\rho^{T_2}$  and fulfilling the angular

momentum algebra, but the form of the operators will in general be different from the above one obtained in the standard basis. From the above observations it follows that the eigensystem of  $\rho^{T_2}$  has the same multiplet structure as  $\rho$  [14] and can therefore be written in the general form

$$\rho^{T_2} = \sum_{K=|S_1-S_2|}^{S_1+S_2} \frac{B(K)}{2K+1} \sum_{K^z=-K}^K |K, K^z\rangle_{00} \langle K, K^z|, \quad (2)$$

where the multiplets are labeled by the value of  $\vec{K}^2 = K(K+1)$  with  $|S_1 - S_2| \leq K \leq S_1 + S_2$  and have degeneracy  $2K+1$ . Again, the real coefficients  $B(K)$  fulfill  $\sum_K B(K) = 1$  (since  $\text{tr}\rho = \text{tr}\rho^{T_2}$ ) but are not necessarily positive. As pointed out by Peres [8], negative  $B(K)$  indicate entanglement in the original state  $\rho$ . The coefficient of the largest multiplet,  $K = S_1 + S_2$ , is given by [14]

$$\frac{B(S_1 + S_2)}{2(S_1 + S_2) + 1} = \langle \pm S_1, \mp S_2 | \rho | \pm S_1, \mp S_2 \rangle \geq 0, \quad (3)$$

where  $|S_1^z, S_2^z\rangle$  are tensor-product eigenstates of  $S_1^z$  and  $S_2^z$ . In particular,  $B(S_1 + S_2)$  is always non-negative and can alternatively be expressed as

$$\frac{B(S_1 + S_2)}{2(S_1 + S_2) + 1} = \text{tr} \left[ \tilde{P}_{\vec{n}}(S_1 + S_2) \rho^{T_2} \right], \quad (4)$$

where  $\tilde{P}_{\vec{n}}(L)$  is the projector onto the subspace with  $\vec{n} \cdot \vec{K} = L$ , and  $\vec{n}$  is an arbitrary unit vector. As it follows from the above multiplet structure, each eigenvalue of  $\rho^{T_2}$  in the subspace with  $\vec{n} \cdot \vec{K} = L + 1 > 0$  occurs also exactly once in the subspace with  $\vec{n} \cdot \vec{K} = L$ . Thus, for  $|S_1 - S_2| \leq K < S_1 + S_2$  the above relation can be generalized to

$$\frac{B(K)}{2K+1} = \text{tr} \left[ \tilde{P}_{\vec{n}}(K) \rho^{T_2} \right] - \left[ \tilde{P}_{\vec{n}}(K+1) \rho^{T_2} \right], \quad (5)$$

where the right hand side can be rewritten as

$$\begin{aligned} & \text{tr} \left[ \left( \tilde{P}_{\vec{n}}(K) - \tilde{P}_{\vec{n}}(K+1) \right) \rho^{T_2} \right] \\ &= \text{tr} \left[ \left( \tilde{P}_{\vec{n}}(K) - \tilde{P}_{\vec{n}}(K+1) \right)^{T_2} \rho \right] \end{aligned} \quad (6)$$

$$= \text{tr} \left[ (P_{\vec{n}}(K) - P_{\vec{n}}(K+1)) \rho \right]. \quad (7)$$

Here  $P_{\vec{n}}(L)$  is the projector onto the subspace with  $\vec{n} \cdot (\vec{S}_1 - \vec{S}_2) = L$ . In the last equation we have used the fact that the projectors  $\tilde{P}_{\vec{n}}(L)$  are polynomials in the operator  $\vec{n} \cdot \vec{K}$  which turns, in the standard basis, into  $\vec{n} \cdot (\vec{S}_1 - \vec{S}_2)$ . However, the expression (7) contains only the spin operators  $\vec{S}_1, \vec{S}_2$  and the density matrix  $\rho$  itself; therefore this expression is independent of any choice of basis,

$$\frac{B(K)}{2K+1} = \text{tr} \left[ (P_{\vec{n}}(K) - P_{\vec{n}}(K+1)) \rho \right]. \quad (8)$$

Hence any separable  $SU(2)$ -invariant density matrix fulfills

$$\text{tr}[(P_{\bar{n}}(K) - P_{\bar{n}}(K+1))\rho] \geq 0 \quad (9)$$

for  $|S_1 - S_2| \leq K < S_1 + S_2$ , while

$$\text{tr}[(P_{\bar{n}}(K) - P_{\bar{n}}(K+1))\rho] < 0, \quad (10)$$

indicates the presence of entanglement in the state  $\rho$ . Thus, when restricting the full space of density operators to the convex submanifold of  $SU(2)$ -invariant states, the operators  $[P_{\bar{n}}(K) - P_{\bar{n}}(K+1)]$ ,  $|S_1 - S_2| \leq K < S_1 + S_2$ , have the properties of entanglement witnesses [10,28]. It is an interesting question whether and, if so, to what extent, one can relax the restriction to  $SU(2)$ -invariant states with this property of the operators  $[P_{\bar{n}}(K) - P_{\bar{n}}(K+1)]$  being unaltered. Note also that the above operators can, by construction, only detect entanglement in  $SU(2)$ -invariant states with negative partial transpose, although these operators do not fulfill the construction recipe of decomposable entanglement witnesses [5].

Moreover, the contributions to the right hand side of Eq. (8) can be expressed as

$$\text{tr}[P_{\bar{n}}(K)\rho] = \sum_{S_1^z - S_2^z = K} \langle S_1^z, S_2^z | \rho | S_1^z, S_2^z \rangle. \quad (11)$$

Thus, the eigenvalues of the partial transpose  $\rho^{T_2}$  are entirely determined by the diagonal elements of  $\rho$  in a basis of tensor-product states of both spins with respect to a common quantization axis. In particular, the relation

$$\begin{aligned} \frac{B(K)}{2K+1} &= \sum_{S_1^z - S_2^z = K} \langle S_1^z, S_2^z | \rho | S_1^z, S_2^z \rangle \\ &- \sum_{S_1^z - S_2^z = K+1} \langle S_1^z, S_2^z | \rho | S_1^z, S_2^z \rangle \end{aligned} \quad (12)$$

provides a convenient way to compute the eigenvalues of  $\rho^{T_2}$  without explicitly solving for the zeros of a characteristic polynomial. Below we shall encounter yet another method to determine the spectrum of  $\rho^{T_2}$  based on sum rules for its eigenvalues.

To gain further insight into the properties of  $\rho^{T_2}$  consider

$$\text{tr}[\vec{K}^2 \rho^{T_2}] = \text{tr}\left[\left(\vec{K}^2\right)^{T_2} \rho\right] \quad (13)$$

$$= \text{tr}\left[\left(\left(\vec{S}_1 - \vec{S}_2\right)^2\right) \rho\right] \quad (14)$$

for  $0 \leq n \leq 2 \min\{S_1, S_2\}$ . In the last equation we have used that the operator  $S_2^y$ , when expressed in the standard basis, changes sign under partial transposition while  $S_2^x$  and  $S_2^z$  remain unaltered. Alternatively, the left hand side of Eq. (13) can also be evaluated using Eq. (2) leading to

$$\left\langle \left(\vec{S}_1 - \vec{S}_2\right)^2 \right\rangle = \sum_{K=|S_1-S_2|}^{S_1+S_2} K(K+1)B(K), \quad (15)$$

where  $\langle \cdot \rangle$  denotes an expectation value with respect to  $\rho$ .

It is instructive to investigate the condition

$$\left\langle \left(\vec{S}_1 - \vec{S}_2\right)^2 \right\rangle \geq (S_1 + S_2)(S_1 + S_2 + 1) \quad (16)$$

which is equivalent to

$$\langle \vec{S}_1 \cdot \vec{S}_2 \rangle < -S_1 S_2 \quad (17)$$

and implies that  $\rho^{T_2}$  has at least one negative eigenvalue, since otherwise we had

$$\begin{aligned} \left\langle \left(\vec{S}_1 - \vec{S}_2\right)^2 \right\rangle &= \sum_{K=|S_1-S_2|}^{S_1+S_2} K(K+1)B(K) \\ &\leq (S_1 + S_2)(S_1 + S_2 + 1) \sum_{K=|S_1-S_2|}^{S_1+S_2} B(K) \\ &= (S_1 + S_2)(S_1 + S_2 + 1). \end{aligned} \quad (18)$$

Thus, the inequalities (16) and (17) are a *sufficient condition* for  $\rho^{T_2}$  having at least one negative eigenvalue, and, in turn, for  $\rho$  being entangled. The latter statement follows also directly from (17), because the right hand side of this inequality represents the minimum value the correlator  $\langle \vec{S}_1 \cdot \vec{S}_2 \rangle$  can attain in a separable state. Therefore, if (17) is fulfilled, the underlying state must be entangled. Note that for general spins  $\vec{S}_1, \vec{S}_2$  the above correlator is bounded by  $-(S_1 + 1)S_2 \leq \langle \vec{S}_1 \cdot \vec{S}_2 \rangle \leq S_1 S_2$  (assuming  $S_1 \geq S_2$ ).

Moreover, the conditions (16) and (17) are also a *necessary criterion* for  $\rho^{T_2}$  having the maximum possible number of negative eigenvalues. Here all  $B(K)$  with  $|S_1 - S_2| \leq K < S_1 + S_2$  are negative, while  $B(S_1 + S_2) =: \bar{B} > 1$  because of the normalization condition  $\sum_K B(K) = 1$ . The assertion is proved as follows,

$$\begin{aligned} \left\langle \left(\vec{S}_1 - \vec{S}_2\right)^2 \right\rangle &\geq (S_1 + S_2 - 1)(S_1 + S_2)(1 - \bar{B}) \\ &\quad + (S_1 + S_2)(S_1 + S_2 + 1)\bar{B} \\ &= (S_1 + S_2 - 1)(S_1 + S_2) \\ &\quad + 2\bar{B}(S_1 + S_2) \\ &\geq (S_1 + S_2)(S_1 + S_2 + 1). \end{aligned} \quad (19)$$

The above considerations can obviously be extended to higher powers of  $\vec{K}^2$ , i.e.  $(\vec{K}^2)^n$  with  $n > 1$ . However, when performing the partial transposition more complicated operator products occur which give rise to additional contributions. E.g. for the next higher powers one finds

$$\left(\left(\vec{K}^2\right)^2\right)^{T_2} = \left(\left(\vec{S}_1 - \vec{S}_2\right)^2\right)^2 + 4\vec{S}_1 \cdot \vec{S}_2 \quad (20)$$

and

$$\begin{aligned} \left( (\vec{K}^2)^3 \right)^{T_2} &= \left( (\vec{S}_1 - \vec{S}_2)^2 \right)^3 - 32 (\vec{S}_1 \cdot \vec{S}_2)^2 \\ &\quad + 4 (3(S_1(S_1 + 1) + S_2(S_2 + 1)) - 4) \vec{S}_1 \cdot \vec{S}_2 \\ &\quad + 8S_1(S_1 + 1)S_2(S_2 + 1) \end{aligned} \quad (21)$$

leading to the additional sum rule

$$\begin{aligned} &\left\langle \left( (\vec{S}_1 - \vec{S}_2)^2 \right)^2 + 4\vec{S}_1 \cdot \vec{S}_2 \right\rangle \\ &= \sum_{K=|S_1-S_2|}^{S_1+S_2} (K(K+1))^2 B(K) \end{aligned} \quad (22)$$

and an analogous relation for  $n = 3$  following from Eq. (21). Eqs. (15), (22), together with the normalization condition  $\sum_K B(K) = 1$ , form a series of sum rules being linear in the coefficients  $B(K)$ . This series can obviously be extended to arbitrary high powers of the spin operators. The number of independent sum rules, however, is in general given by  $2 \min\{S_1, S_2\} + 1$ . Thus, for given  $S_1, S_2$ , the relations arising from  $n = 0, \dots, 2 \min\{S_1, S_2\}$  constitute a linear system of equations which uniquely determines the spectrum of  $\rho^{T_2}$ . Note that the coefficients in this system of equations are of the form  $(K(K+1))^n$ , i.e. the corresponding matrix is of the Vandermonde type with its determinant given by

$$\prod_{\substack{K, L=|S_1-S_2| \\ K > L}}^{S_1+S_2} (K(K+1) - L(L+1)), \quad (23)$$

which is always positive. Such a system of linear equations for the coefficients  $B(K)$  provides an alternative way to compute the eigenvalues of  $\rho^{T_2}$  in terms of spin correlators.

Moreover, using the relation

$$\left\langle (\vec{S}_1 + \vec{S}_2)^2 \right\rangle = \sum_{J=|S_1-S_2|}^{S_1+S_2} J(J+1)A(J) \quad (24)$$

and Eq. (15) one derives the following sum rule

$$\begin{aligned} &2(S_1(S_1 + 1) + S_2(S_2 + 1)) \\ &= \sum_{L=|S_1-S_2|}^{S_1+S_2} L(L+1)(A(L) + B(L)), \end{aligned} \quad (25)$$

where the left hand side is independent of the given state  $\rho$ .

Let us illustrate the above findings on some examples. The simple case when one of the spins, say  $\vec{S}_2$ , has length  $1/2$  was already fully discussed in Ref. [14]. Here one finds

$$B\left(S - \frac{1}{2}\right) = \frac{1}{2S+1} \left( S + 2 \langle \vec{S}_1 \cdot \vec{S}_2 \rangle \right), \quad (26)$$

$$B\left(S + \frac{1}{2}\right) = \frac{1}{2S+1} \left( S + 1 - 2 \langle \vec{S}_1 \cdot \vec{S}_2 \rangle \right) \quad (27)$$

where  $S := S_1$ . Clearly,  $B(S+1/2)$  is always nonnegative (since  $\langle \vec{S}_1 \cdot \vec{S}_2 \rangle \leq S/2$ ), while  $B(S-1/2)$  becomes negative if  $\langle \vec{S}_1 \cdot \vec{S}_2 \rangle < -S/2$ , in accordance with the above results for general spin lengths. Moreover, as shown in Ref. [14], in the case  $S_2 = 1/2$ , there are no entangled states with positive partial transpose, i.e. the Peres-Horodecki criterion for separability is necessary and sufficient.

Next let us consider  $S_2 = 1, S_1 = S \geq 1$ . Here we can use the relations (15), (22) along with the normalization condition to obtain the coefficients  $B(K)$  as

$$\begin{aligned} B(S-1) &= \frac{1}{2S+1} \left( -1 + \langle \vec{S}_1 \cdot \vec{S}_2 \rangle \right. \\ &\quad \left. + \frac{1}{S} \left\langle (\vec{S}_1 \cdot \vec{S}_2)^2 \right\rangle \right), \end{aligned} \quad (28)$$

$$B(S) = 1 - \frac{1}{S(S+1)} \left\langle (\vec{S}_1 \cdot \vec{S}_2)^2 \right\rangle, \quad (29)$$

$$\begin{aligned} B(S+1) &= \frac{1}{2S+1} \left( 1 - \langle \vec{S}_1 \cdot \vec{S}_2 \rangle \right. \\ &\quad \left. + \frac{1}{S+1} \left\langle (\vec{S}_1 \cdot \vec{S}_2)^2 \right\rangle \right). \end{aligned} \quad (30)$$

Again the coefficient of the largest multiplet is of course always nonnegative,  $B(S+1) \geq 0$ , while the conditions for  $B(S-1) < 0$  and  $B(S) < 0$  read

$$1 > \langle \vec{S}_1 \cdot \vec{S}_2 \rangle + \frac{1}{S} \left\langle (\vec{S}_1 \cdot \vec{S}_2)^2 \right\rangle, \quad (31)$$

$$\left\langle (\vec{S}_1 \cdot \vec{S}_2)^2 \right\rangle > S(S+1), \quad (32)$$

respectively. These inequalities generalize the conditions given in Ref. [14] for  $S = 1$  to the case of general spin length  $S$ . Besides, demanding that both  $B(S-1)$  and  $B(S)$  should be negative leads to the necessary condition

$$\langle \vec{S}_1 \cdot \vec{S}_2 \rangle < -S, \quad (33)$$

and it is also easy to explicitly show from the above relations that at least one eigenvalue of  $\rho^{T_2}$  has to be negative if (33) is fulfilled, both in accordance with our earlier general findings.

Alternatively, the coefficients  $B(K)$  characterizing  $\rho^{T_2}$  can be expressed in terms of the quantities  $A(J)$  describing  $\rho$ ,

$$B(S-1) = \frac{2S-1}{2S+1} - \frac{S-1}{S} A(S-1) - \frac{2S-1}{2S+1} \frac{S+1}{S} A(S), \quad (34)$$

$$B(S) = \frac{1}{S+1} - \frac{2S+1}{S(S+1)} A(S-1) + \frac{S-1}{S} A(S), \quad (35)$$

$$B(S+1) = \frac{1}{(2S+1)(S+1)} + \frac{S+2}{S+1} A(S-1) + \frac{2}{2S+1} A(S). \quad (36)$$

Here  $A(S+1)$  has been eliminated via the normalization condition, and the other coefficients can be expressed in terms of spin correlators as follows [14],

$$A(S-1) = \frac{1}{S(2S+1)} \left( -S - (S-1) \langle \vec{S}_1 \cdot \vec{S}_2 \rangle + \left\langle \left( \vec{S}_1 \cdot \vec{S}_2 \right)^2 \right\rangle \right), \quad (37)$$

$$A(S) = 1 - \frac{1}{S(S+1)} \left( \langle \vec{S}_1 \cdot \vec{S}_2 \rangle + \left\langle \left( \vec{S}_1 \cdot \vec{S}_2 \right)^2 \right\rangle \right). \quad (38)$$

Moreover, most recently Breuer has investigated the case  $S_1 = S_2 = 1$  using a different approach and concluded that for this case the PPT criterion is necessary and sufficient, i.e. there are no entangled states with positive partial transpose [22]. This finding also confirms a conjecture raised recently in Ref. [19]. The question whether this is also true for general  $S_1 = S > 1$ ,  $S_2 = 1$  remains open. The approach of Ref. [22] finds linear expressions for the coefficients  $B(K)$  in terms of the  $A(J)$  (in the notation used here). Eqs. (34)-(36) are an example of such a linear relation for the case of  $S_2 = 1$  and general  $S_1 = S \geq 1$ , while the results of Ref. [22] are restricted to equal spin lengths  $S_1 = S_2 \leq 3/2$ .

### III. COMPARISON WITH OTHER ENTANGLEMENT CRITERIA

We now compare the above findings from the PPT criterion with other entanglement criteria. These criteria are generally weaker than the PPT criterion, but have the merit of being very readily applied to  $SU(2)$ -invariant states.

#### A. The reduction criterion and the majorization criterion

The reduction criterion [23,24] states that if a given state  $\rho$  is separable, then the operators

$$\begin{aligned} \rho_1 \otimes \mathbf{1} - \rho \\ \mathbf{1} \otimes \rho_2 - \rho \end{aligned}$$

are also positive, i.e. do not contain any negative eigenvalue. Here  $\rho_{1/2} = \text{tr}_{2/1}(\rho)$  denotes the reduced density matrices of each subsystem. This criterion is in general weaker than the PPT criterion. If one of the subsystems has a Hilbert space of dimension two, however, both criteria are equivalent [24]. Thus, in particular, the reduction criterion is necessary and sufficient for the case of the dimensions  $2 \times 2$  or  $2 \times 3$ .

Applying the reduction criterion to  $SU(2)$ -invariant states is technically very easy since, due to the rotational invariance of these objects, we have

$$\rho_{1/2} = \frac{1}{2S_{1/2} + 1} \mathbf{1}. \quad (39)$$

Thus, the criterion is violated if

$$\frac{A(J)}{2J+1} > \frac{1}{2S_{1/2} + 1} \quad (40)$$

for some  $J$ . If this inequality is fulfilled, the underlying  $SU(2)$ -invariant state  $\rho$  is inseparable. Because  $A(J) \leq 1$  this is only possible for  $J < S_{1/2}$ , which strongly restricts the power of this entanglement criterion as applied to  $SU(2)$ -invariant states [22].

Let us now compare the reduction criterion with the results obtained from the PPT criterion. For the case  $S_1 = S$ ,  $S_2 = 1/2$  the reduction criterion is violated if

$$A(S-1/2) > \frac{2S}{2S+1}, \quad (41)$$

or, using  $A(S-1/2) = (S - 2\langle \vec{S}_1 \cdot \vec{S}_2 \rangle)/(2S+1)$  (cf. Ref. [14]),

$$\langle \vec{S}_1 \cdot \vec{S}_2 \rangle < -\frac{S}{2}. \quad (42)$$

This condition is of course the same as found from the PPT criterion since both criteria are equivalent for this case.

For the case  $S_1 = S \geq 1$ ,  $S_2 = 1$  violation of the reduction criterion leads to the condition

$$A(S-1) > \frac{2S-1}{2S+1}, \quad (43)$$

or, using Eq. (37),

$$-\frac{(S-1)}{S} \langle \vec{S}_1 \cdot \vec{S}_2 \rangle + \frac{1}{S} \left\langle \left( \vec{S}_1 \cdot \vec{S}_2 \right)^2 \right\rangle > 2S. \quad (44)$$

For  $S = 1$  this inequality is the same as the criterion (32). The other inequality (31), however, is not reproduced by the reduction criterion, and for  $S > 1$  the above inequality (44) is a weaker criterion for entanglement than (32). In fact, demanding that  $\rho^{T_2}$  is positive, i.e.  $B(S-1) \geq 0$  and  $B(S) \geq 0$ , one derives from Eqs. (34),(35) the necessary condition

$$A(S-1) \geq \frac{S^2 - (S-1)}{S^2} \frac{2S-1}{2S+1}. \quad (45)$$

Thus, whenever the PPT criterion is unable to detect entanglement in a given state  $\rho$ , the reduction criterion will also fail. As mentioned above, this is a general property [23,24].

Another entanglement criterion related to the reduction criterion is the majorization criterion [25,6]. It states that any separable state  $\rho$  fulfills the inequalities

$$\begin{aligned} \lambda_\rho^\downarrow &\prec \lambda_{\rho_1}^\downarrow \\ \lambda_\rho^\downarrow &\prec \lambda_{\rho_2}^\downarrow \end{aligned}$$

where the vector  $\lambda_\rho^\downarrow$  consists of the eigenvalues of  $\rho$  in decreasing order. The notation  $x \prec y$  means  $\sum_{j=1}^k x_j \leq \sum_{j=1}^k y_j$  for  $k \in \{1, \dots, d\}$ , where the equality holds for  $k = d$ . Here  $d$  is the dimension of the total Hilbert space of the bipartite system, and the vectors  $\lambda_{\rho_{1/2}}^\downarrow$  are extended by zeros in order to make their dimension equal to that of  $\lambda_\rho^\downarrow$ . Obviously the majorization criterion is fulfilled if

$$\frac{A(J)}{2J+1} < \frac{1}{2S_{1/2}+1}. \quad (46)$$

Thus, when applied to SU(2)-invariant states, the majorization criterion is always weaker than the reduction criterion.

## B. Local uncertainty relations

Entanglement criteria based on so-called local uncertainty relations were introduced recently by Hofmann and Takeuchi [26], and by Gühne [27]. This concept is based on the following observation. Let  $\rho = \sum_k p_k \rho_k$ ,  $p_k \geq 0$ ,  $\sum_k p_k = 1$  be a convex combination of some states  $\rho_k$  and let  $M_i$  some set of operators. Then the following inequality holds [26,27]

$$\sum_i \delta^2(M_i)_\rho \geq \sum_k p_k \sum_i \delta^2(M_i)_{\rho_k} \quad (47)$$

where

$$\delta^2(M)_\rho = \langle M^2 \rangle_\rho - \langle M \rangle_\rho^2 \quad (48)$$

and  $\langle \cdot \rangle_\rho$  denotes an expectation value with respect to  $\rho$ . Consider now a bipartite system with operators  $M_i^{(1)}$ ,

$M_i^{(2)}$  acting on one of the subsystems. Then for any separable state  $\rho_k$  one has

$$\delta^2(M_i^{(1)} + M_i^{(2)})_{\rho_k} = \delta^2(M_i^{(1)})_{\rho_k} + \delta^2(M_i^{(2)})_{\rho_k}. \quad (49)$$

Let now  $U_{1/2}$  be the absolute minimum of  $\sum_i \delta^2(M_i^{(1/2)})$  with respect to all possible states of each subsystem. Then any separable state  $\rho$  has to fulfill the inequality

$$\sum_i \delta^2(M_i^{(1)} + M_i^{(2)})_\rho \geq U_1 + U_2. \quad (50)$$

Violation of this inequality is indicative of entanglement in the underlying state  $\rho$ . Note that this observation provides a whole variety of entanglement criteria since the operators  $M_i^{(1/2)}$  are undetermined so far. In circumstances of SU(2)-invariant states, however, it is natural to choose  $M_i^{(1/2)} = S_{1/2}^i$ ,  $i \in \{x, y, z\}$  with  $U_{1/2} = S_{1/2}$  [26]. Then a given SU(2)-invariant state  $\rho$  is entangled if

$$\langle \vec{S}_1 \cdot \vec{S}_2 \rangle < -\frac{1}{2} (S_1^2 + S_2^2) \quad (51)$$

$$= -S_1 S_2 - \frac{1}{2} (S_1 - S_2)^2. \quad (52)$$

The second version of this inequality suggests that this entanglement criterion is strongest if both spins are of the same length,  $S_1 = S_2$ . In this case, the criterion again states that the correlator  $\langle \vec{S}_1 \cdot \vec{S}_2 \rangle$  has to be smaller than its minimum value in any separable state, from which it follows that the underlying state has a negative partial transpose.

The local uncertainty relation of the above form is based on a very natural choice of operators, but provides in general only a quite weak entanglement criterion. For instance, the above spin correlator is bounded from below by  $-(S_1+1)S_2 \leq \langle \vec{S}_1 \cdot \vec{S}_2 \rangle$  (assuming  $S_1 \geq S_2$ ). Thus, the above inequality cannot be fulfilled if  $S_1$  sufficiently exceeds  $S_2$ . We leave it open whether another choice of operators could lead to stronger criteria for inseparability.

## IV. CONCLUSIONS

We have investigated entanglement in SU(2)-invariant bipartite quantum states and have substantially extended previous results on the behavior of such states under partial transposition. The spectrum of the partial transpose of a given SU(2)-invariant density matrix  $\rho$  is entirely determined by the diagonal elements of  $\rho$  in a basis of tensor-product states of both spins with respect to a common quantization axis. We have constructed a set of operators which act as entanglement witnesses on SU(2)-invariant states, and we have derived sufficient criterion for  $\rho^{T_2}$  having at least one negative eigenvalue in terms of a simple spin correlator. The same condition is a necessary criterion for the partial transpose to have the maximum number of negative eigenvalues. Moreover, we have

presented a series of sum rules which uniquely determine the eigenvalues of the partial transpose in terms of a system of linear equations. Finally we have compared our findings with other entanglement criteria including the reduction criterion, the majorization criterion, and the recently proposed local uncertainty relations.

The key challenge for future investigations of  $SU(2)$ -invariant states (or states being invariant under other transformation groups) is certainly to determine to what extent the PPT criterion is necessary *and sufficient*. A possible route toward this goal could be given by the methods developed in the recent preprint [22]. This approach, however, is so far limited to the case of equal spin lengths  $S_1 = S_2 \leq 3/2$ .

- [23] M. Horodecki and P. Horodecki, Phys. Rev. A **59**, 4206 (1999).
- [24] N. Cerf, C. Adami, and R. M. Gingrich, Phys. Rev. A **60**, 898 (1999).
- [25] M. A. Nielsen and J. Kempe, Phys. Rev. Lett, **86**, 5184 (2001).
- [26] H. F. Hofmann and S. Takeuchi, Phys. Rev. A **68**, 032103 (2003).
- [27] O. Gühne, Phys. Rev. Lett. **92**, 117903 (2004); O. Gühne and M. Lewenstein, quant-ph/0409091.
- [28] B. M. Terhal, Phys. Lett. **271**, 319 (2000).

- 
- [1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
  - [2] E. Schrödinger, Naturwiss. **23**, 807 (1935).
  - [3] For a general overview see e.g. M. A. Nielsen and I. L. Chuang, “*Quantum Computation and Quantum Information*”, Cambridge University Press, 2000.
  - [4] M. Lewenstein, D. Bruß, J. I. Cirac, M. Kus, J. Samsonowicz, A. Sanpera, and R. Tarrach, J. Mod. Opt. **47**, 2841 (2000).
  - [5] B. M. Terhal, Theor. Comput. Sci. **287**, 313 (2002).
  - [6] D. Bruß, J. Math. Phys. **43**, 4237 (2002).
  - [7] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
  - [8] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
  - [9] We adopt the somewhat sloppy but common jargon that an operator is said to be positive if it does not have any negative eigenvalue.
  - [10] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 8 (1996).
  - [11] P. Horodecki, Phys. Lett. A **232**, 333 (1997).
  - [12] K. G. H. Vollbrecht and R. F. Werner, Phys. Rev. A **64**, 062307 (2002).
  - [13] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
  - [14] J. Schliemann, Phys. Rev. A **68**, 012309 (2003).
  - [15]  $SU(2)$ -invariant states were also studied by B. Hendriks and R. F. Werner [B. Hendriks, Diploma thesis, University of Braunschweig, Germany, 2002].
  - [16] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature **416**, 608 (2002).
  - [17] T. J. Osborne and M. A. Nielsen, Phys. Rev. A **66**, 032110 (2002).
  - [18] F. Verstraete, M. Popp, and J. I. Cirac, Phys. Rev. Lett. **92**, 027901 (2004).
  - [19] X. Wang, H. Li, Z. Sun, and Y.-Q. Li, quant-ph/0501032.
  - [20] For an guide to further recent publications see also the references given in Ref. [19].
  - [21] G. A. Durkin, C. Simon, J. Eisert, and D. Bouwmeester, Phys. Rev. A **70**, 062305 (2004).
  - [22] H.-P. Breuer, quant-ph/0503079.